

# Gauge-Invariant Average of Einstein Equations for finite Volumes

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For the study of cosmological backreaction an averaging procedure is required. In this work a covariant and gauge invariant averaging formalism for finite volumes will be developed. This averaging will be applied to the scalar parts of Einstein's equations. For this purpose dust as a physical laboratory will be coupled to the gravitating system. The goal is to study the deviation from the homogeneous universe and the impact of this deviation on the dynamics of our universe. Fields of physical observers are included in the studied system and used to construct a reference frame to perform the averaging without a formal gauge fixing. The derived equations resolve the question whether backreaction is gauge dependent.

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## I. INTRODUCTION

Large scale isotropy and homogeneity is the most fundamental assumption of modern cosmology. Those assumptions make it possible to solve Einstein's equations, which reduce to two differential equations with only one parameter, the scale factor. To test this hypothesis and to estimate the effect of the deviations from homogeneity and isotropy an averaging formalism was developed in [1–3]. It has been criticized in [4, 5] that this approach relies on an a priori fixed frame and is thus gauge dependent. An alternative has been suggested which is valid for infinite spacial volumes. In this work we use a physical reference frame, the frame of comoving dust, to perform the averaging in a manifestly gauge independent way. Our formalism is applicable to finite volumes. At the end we compare our result to the equations obtained in [2, 3] and show that they are equivalent in the frame of the introduced pressure-less fluid.

## II. GAUGE INVARIANT BUCHERT EQUATIONS

Starting from local inhomogeneous Einstein equations an averaging procedure is set up to derive effective equations which quantitatively measure the deviation from the Friedmann equations which describe the dynamics of a homogeneous and isotropic space-time. To avoid the question of coordinate fixing and hence gauge dependence a system of freely falling observers will be introduced and used to perform the averaging. It is important to mention the compatibility of the dust as coordinate system with a deformed FRW universe, as it respects the fundamental symmetries of the background. We know that our universe used to be homogeneous to a very high precision and hence was described well by the FRW universe. However, nowadays there are large inhomogeneities locally

present in the universe. Nevertheless, observations of the large scale structure suggest that the underlying geometry is not drastically different from an homogeneous and isotropic space on horizon scales. Therefore, it is sensible to use a laboratory which does not destroy the FRW space-time, since it has the appropriate symmetries. The dust we are going to use fulfils this criterion.

### *The model*

We consider a space-time filled with matter and demand that the energy density of this matter should never vanish. This assumption allows to avoid the problem of seemingly indeterministic behaviour of General Relativity (GR), known as the hole problem [6]. Demanding a non vanishing matter content we are able to give physical meaning to every point in space-time. This is also a reasonable assumption on physical grounds, since even the voids contain some amounts of cosmic dust or radiation. Furthermore, it is speculated that there is matter present in the universe, which interacts only (or almost only) with gravity. The dark matter is considered to be “cold” i.e. non-relativistic in the standard model of cosmology, hence it can be well described by dust. Thus, our model can be viewed as a realistic cosmological model incorporating the cold dark matter.

The distribution of matter in our model though is not homogeneous, since it is supposed to represent a universe perturbed about the FRW background symmetry. This is the difference to the approach in [4, 5], we allow for matter to exist which is distributed inhomogeneously and hence it is possible to find physical fields with a space-like gradient.

Within our model the energy momentum tensor can be decomposed in the way described below, since we have demanded for the energy density never to vanish. We can view this in two ways. Either the dust of low energy is formally separated from the total energy momentum tensor, or it is added to it, which would not affect the system strongly.

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$$T^{\mu\nu} = T_{\text{dust}}^{\mu\nu} + \tilde{T}^{\mu\nu}, \quad (1)$$

where the  $T_{\text{dust}}^{\mu\nu} = \epsilon u^\mu u^\nu$  is the dust energy momentum with infinitesimal  $\epsilon$  and  $\tilde{T}^{\mu\nu}$  is the energy momentum tensor of the rest of the system.

Consequently the Lagrangian of the theory is as follows

$$L = L_{\text{gravity}} + \tilde{L} + L_{\text{dust}}. \quad (2)$$

The gravitational Lagrangian is the Ricci scalar multiplied by the square root of the metric determinant. The dust Lagrangian is

$$L = \sqrt{-g} \epsilon \{v^\mu v^\nu g_{\mu\nu} + 1\}. \quad (3)$$

With the following eight fields,  $\epsilon$  being the energy density,  $T$  eigentime,  $W_l$  velocity field and  $Z^k$  coordinate field, where  $k, l \in \{1, 2, 3\}$

$$v^\mu = -\partial^\mu T + W_k \partial^\mu Z^k, \quad (4)$$

$T$  has a time-like gradient and the  $Z^k$ 's have obviously space-like gradients. These fields will be used for the construction of a window function which transforms as a scalar. We will restrict our analysis to the Lagrangian formalism and not be concerned with the form of the physical Hamiltonian. On the other hand we know that a physical Hamiltonian exists, since the system with dust deparametrizes, as has been shown in [7–9].

### A. Time

The eigentime of the dust particles is a map into the dust-time manifold

$$T : (\mathbb{R} \times \Sigma) \rightarrow \mathcal{T}. \quad (5)$$

At each fixed value  $T = \tau$ ,  $T$  is a scalar w.r.t. coordinate changes on the manifold  $\mathcal{M}$  in the sense

$$\tilde{T}(X) = T(\tilde{X}), \quad (6)$$

where  $\tilde{X} = f^{-1}(X)$ . On the dust-time manifold  $T$  is trivially a scalar. This will be scalar with a space-like gradient which will be used for the construction of the window function. We take the eigentime  $T$  of the dust particles as the scalar field which generates the time flow. Therefore the unit time-like vector is

$$n^\mu = \frac{-\partial^\mu T}{\sqrt{-\partial_\mu T \partial^\mu T}}. \quad (7)$$

The foliation is defined as the hypersurfaces on which  $T$  takes constant values. It is the physical time of the dust field. This defines a time-space split with the projector orthogonal to the time flow

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \quad (8)$$

### B. Space

The dust is a collection of geodesic observers, freely falling on  $\mathcal{M}$ . Each particle carries a label, at any instant of time  $\tau$  a particle has a position, so this position of the labeled particle is a physical value and hence a Dirac observable. Taking the continuum limit, while keeping the particle density constant we obtain a field, where the indices of the particles can be viewed now as three dimensional coordinates  $Z^i$ ,  $i \in \{1, 2, 3\}$ .  $Z$  is a map from the hypersurface of constant time into the dust-space manifold of constant  $T = \tau$

$$Z : \Sigma \rightarrow S(\tau). \quad (9)$$

Now we define the quantity:  $\bar{Z}^i(T(\tau)) := Z^i(\tau)$ . The  $\bar{Z}^i$  are for every fixed pair of values  $Z^i$  and  $T$  scalars on  $\mathcal{M}$ , (omit the bar in the coming discussion)

$$\tilde{Z}^k(X) = Z^k(\tilde{X}), \quad (10)$$

where  $\tilde{X} = f^{-1}(X)$ . On the other hand they are three vectors on  $S$ . To construct a quantity which is a scalar on  $\mathcal{M}$  and  $S$ , more work is required. The basic idea is to construct a scalar on  $S$  out of the coordinate fields of the dust, a quantity like  $Z_k Z^k$ . Therefore we need the induced metric on the hypersurfaces  $h_{ij}$ . To get this object we pull back the orthogonal projector  $h_{\mu\nu}$  on the hypersurface of constant  $T$  denoted by  $\Sigma$ . To stay general we introduce coordinates on  $\Sigma$  called  $x^a$  which represent a map

$$Z^{-1} : S(\tau) \rightarrow \Sigma. \quad (11)$$

Furthermore, the coordinates on  $\mathcal{M}$  are denoted by  $X^\mu(x^a)$  and represent a map

$$X : \Sigma \rightarrow \mathcal{M}. \quad (12)$$

The combined map which we are going to use for the pull-back is  $\rho$ , defined as

$$\rho := X \circ Z^{-1} \Rightarrow \rho : S(\tau) \rightarrow \mathcal{M}. \quad (13)$$

We can visualize the interrelations as follows

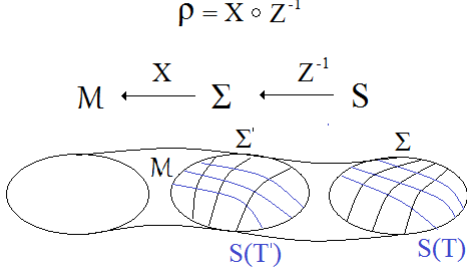


FIG. 1: This pictogram shows the falling of the geodesic observers on the manifold  $\mathcal{M}$ , the hypersurfaces of constant dust-time are denoted by  $\Sigma$  and  $\Sigma'$ . While the map  $Z^{-1}$  goes from the dust-space to the space of internal coordinates of the hypersurface of constant dust-time, the map  $X$  goes from this internal space to the manifold. Therefore, the combination of both maps goes from the dust-space slice to the full space-time manifold and is used for the pull-back.

With the pull backs of the maps given as

$$X_* = \frac{\partial X^\mu}{\partial x^a} =: X_a^\mu, \quad (14)$$

$$Z_*^{-1} = \frac{\partial x^i}{\partial Z^j} =: (Z^{-1})_j^a, \quad (15)$$

$$\rho_* = \frac{\partial X^\mu}{\partial Z^j} = X_a^\mu (Z^{-1})_j^a. \quad (16)$$

The choice of the  $X$  and  $Z^{-1}$  maps corresponds to the gauge choice. Later we will evaluate the result in the the so called ADM gauge<sup>1</sup> where  $\rho = X$  and  $Z^{-1} = \mathbb{I}$ , therefore  $\Sigma = S$ . After defining the maps we come back to the main goal and pull-back on the dust space

$$\begin{aligned} h_{ij}(Z) &= \rho_* h_{\mu\nu} = X_a^\mu (Z^{-1})_i^a X_b^\nu (Z^{-1})_j^b h_{\mu\nu} = \\ &= \frac{\partial X^\mu}{\partial Z^i} \frac{\partial X^\nu}{\partial Z^j} h_{\mu\nu} =: X_i^\mu X_j^\nu h_{\mu\nu} = X_i^\mu X_j^\nu g_{\mu\nu} \\ &+ X_i^\mu X_j^\nu n_\mu n_\nu = X_i^\mu X_j^\nu g_{\mu\nu} = X_a^\mu (Z^{-1})_i^a X_b^\nu (Z^{-1})_j^b g_{\mu\nu}. \end{aligned} \quad (17)$$

Here the property has been used that  $n^\mu$  is orthogonal to the tangents of  $S$ :  $\partial_i X^\mu$ , i.e.  $X_i^\mu n_\mu = 0 \forall i$ .

Note: For the construction we have used the orthogonality of the time flow to the spatial hypersurfaces. To ensure this we can pick the dust velocity field  $W_k$  to vanish. This is simply assuming that the dust we consider

is rotation free, which is reasonable for all practical purposes.

The constructed quantity commutes with the 3-diffeomorphism constraints and in the next step we construct the observable, using the dynamic equation with  $t$  being  $X^0$  from the ADM split:

$$\frac{\partial h_{ij}(Z)}{\partial t} = \{h_{ij}(Z), H(f)\}, \quad (18)$$

here  $H(f) = \int d^3x H_0(x) f(x)$  with  $f$  some scalar density and  $H_0$  the total Hamiltonian constraint of the system. Even if an analytic solution to this equation can not be obtained, we can set initial values by field redefinition and perform a series expansion. The solution of this equation  $h_{ij}(Z, t)$  is used to define the observable  $\bar{h}_{ij}(Z, T)$

$$\bar{h}_{ij}(Z, T(Z, t)) := h_{ij}(Z, t). \quad (19)$$

This quantity commutes for fixed  $T$  and  $Z$  with all constraints and is hence an observable. In the following we will omit the bar for simplicity. Now we found the tensor quantity on  $S$  which is needed for the contraction with the vectors and can write

$$Z^2 = Z^k h_{kl} Z^l =: B. \quad (20)$$

The dust vectors are contracted with a tensor on the dust manifold and one obtains a physical scalar on  $\mathcal{M}$  and on  $S$ . It transforms as

$$\tilde{B}(X) = B(\tilde{X}), \quad (21)$$

and

$$\tilde{B}(x, T) = B(\tilde{x}, T), \quad (22)$$

where on  $\mathcal{M}$ :  $\tilde{X} = f^{-1}(X)$  and correspondingly on  $S$ :  $\tilde{x} = f^{-1}(x)$ .

We check the transformation properties below.

- The induced metric on the hypersurface  $X_a^\mu (Z^{-1})_i^a X_b^\nu (Z^{-1})_j^b g_{\mu\nu}$  is a scalar under coordinate changes on the manifold by construction, an explicit calculation shows it in components:

$$\begin{aligned} \tilde{h}_{ij}(X) &= \frac{\partial}{\partial Z^i} \frac{\partial \tilde{X}^\mu}{\partial X^\lambda} X^\lambda \frac{\partial X^\lambda}{\partial \tilde{X}^\mu} \frac{\partial}{\partial Z^j} \frac{\partial \tilde{X}^\nu}{\partial X^\rho} X^\rho \frac{\partial X^\rho}{\partial \tilde{X}^\nu} g_{\lambda\rho} \quad (23) \\ &= \frac{\partial X^\lambda}{\partial Z^i} \frac{\partial X^\rho}{\partial Z^j} g_{\lambda\rho} = h_{ij}(\tilde{X}). \end{aligned}$$

<sup>1</sup> The Arnowit, Deser, Misner (ADM) formalism was developed to study the Hamiltonian evolution of a gravitating system [10].

- The constructed quantity  $B = Z^2$  is a scalar on  $S$  since the 3-vectors  $Z^k$  are contracted with a rank two tensor on  $S$ . Again explicit calculation shows:

$$\begin{aligned} B(\vec{Z}) &= Z^k h_{kl} Z^l = \left( \frac{\partial Z^k}{\partial x^a} x^a \right) \tilde{h}_{kl} \left( \frac{\partial Z^l}{\partial x^b} x^b \right) = \\ &= \left( \frac{\partial Z^k}{\partial x^a} x^a \right) \left( h_{ab} \frac{\partial x^a}{\partial Z^k} \frac{\partial x^b}{\partial Z^l} \right) \left( \frac{\partial Z^l}{\partial x^b} x^b \right) = x^a h_{ab} x^b = B(\vec{x}). \end{aligned} \quad (24)$$

- The connecting relation which makes the whole object invariant under coordinate transformations on the manifold is that the  $Z^k$  s are the dust labels and hence scalars on  $\mathcal{M}$  as discussed above:

$$\tilde{Z}^k(X) = Z^k(\tilde{X}), \quad (25)$$

where  $\tilde{X} = f^{-1}(X)$ .

Now we have shown that  $B = Z^2$  is a scalar on  $\mathcal{M}$  and on the dust manifold  $S$ . Since the  $Z^2$  is a scalar on  $S$  we can express it in arbitrary coordinates  $x^a$  on  $\Sigma$ , the ADM split hypersurface of constant  $T$ . This coordinate representation is also convenient for the Window function which will be discussed in the following section.

### C. Window function

Having constructed physical scalars we write the window function as

$$W_\Omega = \delta(T(x) - T_0) \sqrt{-\partial_\mu T \partial^\mu T} \Theta(r_0 - B(x)), \quad (26)$$

with  $\delta(x)$  is a Dirac delta function and  $\Theta(x)$  the step function. The Window function constructed in (26) is a scalar on  $\mathcal{M}$  and  $S$  i.e. it transforms as

$$W \rightarrow \tilde{W}(x) = W(f^{-1}(x)). \quad (27)$$

Now the functional is set up in the following way

$$\begin{aligned} F(\Omega) &= \\ &= \int_{\mathcal{M}_4} \sqrt{-g(x)} \sqrt{-\partial_\mu T \partial^\mu T} \Theta(r_0 - B(x)) \delta(T_0 - T(x)) d^4x. \end{aligned} \quad (28)$$

The averaging functional of a quantity  $S$  can be defined using Eqn. (28), and simplified integrating out the delta function leading to

$$\begin{aligned} \langle S \rangle_{\{A, r_0\}} &= \frac{F(S, \Omega)}{F(1, \Omega)} = \frac{\int_{\mathcal{M}_4} \sqrt{-g(x)} \sqrt{-\partial_\mu T \partial^\mu T} S(x) \Theta(r_0 - B(x)) \delta(T_0 - T(x)) d^4x}{\int_{\mathcal{M}_4} \sqrt{-g(x)} \sqrt{-\partial_\mu T \partial^\mu T} \Theta(r_0 - B(x)) \delta(T_0 - T(x)) d^4x} = \\ &= \frac{\int_{\Sigma_0} \int_{\mathbb{R}} N \sqrt{h(\vec{x})} \sqrt{-\partial_\mu T \partial^\mu T} S(x) \Theta(r_0 - B(x)) \delta(T_0 - T(x)) d^3x dT}{\int_{\Sigma_0} \int_{\mathbb{R}} N \sqrt{h(\vec{x})} \sqrt{-\partial_\mu T \partial^\mu T} \Theta(r_0 - B(x)) \delta(T_0 - T(x)) d^3x dT} = \\ &= \frac{\int_{\Sigma_0} N \sqrt{h(x)} \sqrt{-\partial_\mu T \partial^\mu T}|_{T=T_0} S(x) \Theta(r_0 - B(x)) d^3x}{\int_{\Sigma_0} N \sqrt{h(x)} \sqrt{-\partial_\mu T \partial^\mu T}|_{T=T_0} \Theta(r_0 - B(x)) d^3x}, \end{aligned} \quad (29)$$

where  $\Sigma_0$  is the hypersurface on which  $T(x) = T_0$ ,  $h$  the determinant of the induced spatial metric discussed above and  $N$  the lapse function for a general not synchronous gauge. The average is manifestly a gauge independent quantity, since  $T(x)$  and  $B(x)$  are scalars under  $\text{Diff}(\mathcal{M})$  and also  $\text{Diff}(S)$  and so the window function is also a scalar, leading to the following transformation property of the averaging functional

$$\begin{aligned} F(S, \Omega) &\rightarrow \tilde{F}(\tilde{S}, \Omega) = \int_{\mathcal{M}_4} \sqrt{-\tilde{g}(x)} \tilde{S}(x) \tilde{W}_\Omega(x) d^4x \\ &= \int_{\mathcal{M}_4} \left| \frac{\partial x}{\partial f} \right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))} S(f^{-1}(x)) W_\Omega(f^{-1}(x)) d^4x, \end{aligned} \quad (30)$$

introducing  $\hat{x} = f^{-1}(x)$  we obtain

$$\tilde{F}(\tilde{S}, \Omega) = \int_{\mathcal{M}_4} \sqrt{-g(\hat{x})} \tilde{S}(\hat{x}) W_\Omega(\hat{x}) d^4\hat{x} = F(S, \Omega). \quad (31)$$

The gauge functional is obviously gauge invariant and hence the average of  $S$  is an observable.

Equipped with this functional we can address the problem of averaging the scalar parts of Einstein's equations of the model of the universe presented above, where the coordinate system, the dust, is included into the studied system. We will perform a similar analysis as in [4], but in a more general way, since our formalism applies also to finite volumes.

### D. Time derivative of the average

To average the dynamic equations of our model it is also necessary to compute time derivatives of the averaged quantities. In this case the time derivative means partial derivative w.r.t  $T_0$  the eigentime of the homogeneous dust used for the deparametrisation. Let us begin with the derivative of  $F(S, \Omega)$ . The strategy will be to choose certain coordinates in which the computation will be easier and then to generalize our result to an arbitrary frame.

The eigentime  $T$  is the function that defines the space as hypersurfaces on which it assumes a constant value, therefore, it is spatially homogeneous. In this case we can choose coordinates, in future referred to as ADM

coordinates, such that:  $n_\mu = N(-1, 0, 0, 0)$  and  $n^\mu = \frac{1}{N}(1, -N^i)$  where  $N$  is the lapse and  $N^i$  the shift. The metric reads:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (32)$$

Furthermore, since we have set the velocity field  $W^k$  of the coordinate dust to zero, which implies the orthogonality of  $n^\mu$  to the spatial hypersurfaces, the shift vectors vanish. In this coordinates we compute the time derivative. The partial derivative commutes past the integral and the derivative of the  $\delta$  distribution we understand in the weak sense

$$\begin{aligned} \frac{\partial F(S, \Omega)}{\partial T_0} &= - \int_{\mathcal{M}_4} \sqrt{-g(x)} \sqrt{-\partial_\mu T \partial^\mu T} S(x) \Theta(r_0 - B(x)) \frac{\partial}{\partial T} \{\delta(T_0 - T(x))\} d^4x \\ &= - \int_{\mathcal{M}_4} \sqrt{-g(x)} \sqrt{-\partial_\mu T \partial^\mu T} S(x) \Theta(r_0 - B(x)) (\partial_0 T)^{-1} \partial_0 \{\delta(T_0 - T(x))\} d^4x. \end{aligned} \quad (33)$$

In the coordinates chosen we can read off from the form of  $n_\mu$  the only component of  $\partial_\mu T$  which is non-zero:

$\partial_0 T$  and  $-\partial_\mu T g^{\mu\nu} \partial_\nu T = -(\partial_0 T)^2 g^{00}$  with  $g^{00} = -N^{-2}$ . Thus, Eqn. (33) simplifies to

$$\begin{aligned} \frac{\partial F(S, \Omega)}{\partial T_0} &= - \int_{\mathcal{M}_4} \sqrt{-g(x)} S(x) \Theta(r_0 - B(x)) \sqrt{-g^{00}} \partial_0 \{\delta(T_0 - T(x))\} d^4x \\ &= \int_{\mathcal{M}_4} \partial_0 \left\{ \sqrt{-g(x)} S(x) \Theta(r_0 - B(x)) \sqrt{-g^{00}} \right\} \delta(T_0 - T(x)) d^4x \\ &= \int_{\mathcal{M}_4} \sqrt{|h|} \{ \Theta(r_0 - B(x)) (N\theta S + \partial_0 S) - \delta(r_0 - B(x)) S \partial_0 B(x) \} \delta(T_0 - T(x)) d^4x. \end{aligned} \quad (34)$$

Here  $h = \det(h_{ij})$  and  $\theta = \frac{1}{N} \partial_0 \log(\sqrt{h})$ , since  $\theta_{ij} = \frac{1}{2} \mathcal{L}_n h_{ij} = \partial_T h_{ij} = \frac{1}{N} \partial_0 h_{ij}$  in this gauge and the trace  $\theta = \frac{1}{N} h^{ij} \dot{h}_{ij} = \frac{1}{N} \partial_0 \log(\sqrt{h})$ . Having this, we can

rewrite the expression in a covariant way, so that it reduces to our result when the ADM coordinates are fixed

$$\frac{\partial F(S, \Omega)}{\partial T_0} = F \left( \frac{\partial_\mu T \partial^\mu S}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega \right) + F \left( \frac{S\theta}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega \right) - 2 F \left( \frac{\partial_\mu T \partial^\mu B}{\sqrt{-\partial_\mu T \partial^\mu T}} S \delta(r_0 - B), \Omega \right). \quad (35)$$

The last term vanishes in case  $n^\mu \partial_\mu B = 0$  i.e. the spatial coordinates do not depend on the time variable. This is indeed the case in our model since  $n^\mu$  is orthogonal

to the space such that  $\nabla_n Z^k = 0$  which implies the above relation. Hence the time derivative reads

Now we can calculate the time derivative of the average of a scalar

$$\frac{\partial F(S, \Omega)}{\partial T_0} = F \left( \frac{\partial_\mu T \partial^\mu S}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega \right) + F \left( \frac{S \theta}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega \right). \quad (36)$$

$$\begin{aligned} \frac{\partial \langle S, \Omega \rangle}{\partial T_0} &= \frac{\partial F(S, \Omega)}{\partial T_0} \frac{1}{F(1, \Omega)} - \langle S, \Omega \rangle \frac{1}{F(S, \Omega)} \frac{\partial F(1, \Omega)}{\partial T_0} = \\ &= \left( F(\partial_{T_0} S, \Omega) + F \left( \frac{N S \theta}{\partial_0 T}, \Omega \right) \right) \frac{1}{F(1, \Omega)} - \langle S, \Omega \rangle \frac{1}{F(1, \Omega)} F \left( \frac{N \theta}{\partial_0 T}, \Omega \right) = \\ &= \langle \partial_{T_0} S, \Omega \rangle + \left\langle \frac{N S \theta}{\partial_0 T}, \Omega \right\rangle - \langle S, \Omega \rangle \left\langle \frac{N \theta}{\partial_0 T}, \Omega \right\rangle. \end{aligned} \quad (37)$$

Covariantly expressed we obtain the general form of the Buchert, Ehlers commutation rule, [3]:

$$\frac{\partial \langle S, \Omega \rangle}{\partial T_0} = \left\langle \frac{\partial_\mu T \partial^\mu S}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega \right\rangle + \left\langle \frac{S \theta}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega \right\rangle - \langle S, \Omega \rangle \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega \right\rangle. \quad (38)$$

### E. The effective scale factor

In order to be interpreted as a domain dependent scale factor analogous to the  $a$  of the FRW model, a quantity has to be considered which is gauge independent. As any physical observable must not depend on the coordinates chosen.

To accomplish this we use the gauge independent averaging functional to define the quantity  $s$  to be the effective scale factor

$$s = \left( \frac{F(1, \Omega)_{T_0}}{F(1, \Omega)_0} \right)^{\frac{1}{3}} \quad (42)$$

leading to

$$\frac{\partial s}{\partial T_0} = \frac{1}{3} (s^3)^{-\frac{2}{3}} \frac{1}{F_0} \frac{\partial F(1, \Omega)_{T_0}}{\partial T_0} = \frac{1}{3} s \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle. \quad (43)$$

$$\frac{1}{s} \frac{\partial s}{\partial T_0} := \frac{1}{3} \frac{1}{F(1, \Omega)} \frac{\partial F(1, \Omega)}{\partial T_0}. \quad (39)$$

Using (36) one sees that

$$\frac{\partial F(1, \Omega)}{\partial T_0} = F \left( \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}}, \Omega \right), \quad (40)$$

and hence

$$\frac{1}{s} \frac{\partial s}{\partial T_0} = \frac{1}{3} \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle. \quad (41)$$

*Remark:*

A more pictorial way to describe this, is to write

We see from this calculation that  $s$  has indeed a property of a scale factor, but its definition above circumvent the necessity of introducing a normalization at a certain initial time point.

### III. EINSTEIN'S EQUATIONS

We will derive the expressions for the averaged Hamilton constraint density and the Raychaudhuri equation, the important point of the derivations is that in our set up the  $n^\mu$ , which defines the time flow of the coordinate dust, is already present in the model and is not chosen arbitrarily. The time arises natural and there is no need of artificial gauge fixing. Thus, we describe the time flow without formally choosing a gauge.

### A. Gauge independent averaging

*Normal projection of the equations of motion:*

The Einstein tensor multiplied with the time flow vectors (which is equivalent to the Hamiltonian constraint density) gives

$$G^{\mu\nu} n_\mu n_\nu = \frac{1}{2} R_s + \frac{1}{3} \theta^2 - \sigma^2, \quad (44)$$

where the expansion rate and the shear are defined w.r.t  $n_\mu$ , the  $R_s$  is the spatial curvature associated with the projector on the hypersurface orthogonal to  $n_\mu$ .

On the other hand one has from the matter contribution

$$\begin{aligned} T^{\mu\nu} n_\mu n_\nu &= (T_{\text{dust}}^{\mu\nu} + \tilde{T}^{\mu\nu}) n_\mu n_\nu \\ T_{\text{dust}}^{\mu\nu} n_\mu n_\nu &= \epsilon (v_\mu n^\mu)^2 = \epsilon (-\partial_\mu T n^\mu)^2 = \epsilon (-\partial_\mu T \partial^\mu T), \end{aligned} \quad (45)$$

with  $v^\mu = \sqrt{-\partial_\mu T \partial^\mu T} n^\mu$  since we said that the velocity field  $W^k$  vanishes. Now the Hamiltonian constraint density can be averaged. For normalization divide it first by  $-3 \partial_\mu T \partial^\mu T$  apply the averaging functional introduced above and insert the identity by adding and subtracting the average of  $\theta$  squared we get

$$\left\langle \frac{1}{6} \frac{R_s}{-\partial_\mu T \partial^\mu T} + \frac{1}{9} \frac{\theta^2}{-\partial_\mu T \partial^\mu T} - \frac{1}{3} \frac{\sigma^2}{-\partial_\mu T \partial^\mu T} \right\rangle + \frac{1}{9} \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle^2 - \frac{1}{9} \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle^2 = \frac{8\pi G_N}{3} \left\langle \frac{\tilde{T}^{\mu\nu} n_\mu n_\nu}{-\partial_\mu T \partial^\mu T} + \epsilon \right\rangle, \quad (46)$$

using

$$\left( \frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2 = \frac{1}{9} \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle^2, \quad (47)$$

and defining the backreaction

$$\begin{aligned} Q_D &= \frac{1}{3} \left( \left\langle \frac{\theta^2}{-\partial_\mu T \partial^\mu T} \right\rangle - \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle^2 \right) \\ &\quad - 2 \left\langle \frac{\sigma}{-\partial_\mu T \partial^\mu T} \right\rangle, \end{aligned} \quad (48)$$

we rewrite (46) as

$$\begin{aligned} \left( \frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2 &= -\frac{1}{6} \left\langle \frac{R_s}{-\partial_\mu T \partial^\mu T} \right\rangle - \frac{1}{6} Q_D \\ &\quad + \frac{8\pi G_N}{3} \left\langle \frac{\tilde{T}^{\mu\nu} n_\mu n_\nu}{-\partial_\mu T \partial^\mu T} + \epsilon \right\rangle. \end{aligned} \quad (49)$$

This is the effective equation describing the velocity of the expansion of the domain dependent scale factor. It is the analogue to the first Friedmann equation. With the definitions comparable to those in [3] we can write the cosmological energy balance:

$$H_D := \frac{1}{s} \frac{\partial s}{\partial T_0}, \quad (50)$$

$$\Omega_m := \frac{8\pi G_N}{3 H_D^2} \left\langle \frac{\tilde{T}^{\mu\nu} n_\mu n_\nu}{-\partial_\mu T \partial^\mu T} \right\rangle, \quad (51)$$

$$\Omega_\epsilon := \frac{8\pi G_N}{3 H_D^2} \langle \epsilon \rangle, \quad (52)$$

$$\Omega_k := -\frac{\langle R \rangle}{6 H_D^2}, \quad (53)$$

$$\Omega_Q := -\frac{Q_D}{6 H_D^2}, \quad (54)$$

$$\Omega_k + \Omega_Q + \Omega_\epsilon + \Omega_m = 1. \quad (55)$$

From the energy balance equation we see that the backreaction term  $\Omega_Q$  as well as the coordinate mass term  $\Omega_\epsilon$  enter the balance and the backreaction term opens the possibility of contributing to an effective cosmological constant term.

*Raychaudhuri's equation:*

Combining the Hamiltonian constraint density with the trace of the Einstein equations projected orthogonal to the time flow, one gets

Now we will need the second time derivative of the scale factor s:

$$R_{\mu\nu}n^\mu n^\nu = T_{\mu\nu}h^{\mu\nu} - \frac{1}{2}T, \quad (56)$$

which is equivalent to

$$\begin{aligned} -n^\mu \nabla_\mu \theta &= 2\sigma^2 + \frac{1}{3}\theta^2 - \nabla^\nu (n^\mu \nabla_\mu n_\nu) \\ &+ (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)n^\mu n^\nu. \end{aligned} \quad (57)$$

In our model the last term gives

$$\begin{aligned} (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)n^\mu n^\nu &= T_{\mu\nu}^{dust}n^\mu n^\nu + \tilde{T}_{\mu\nu}n^\mu n^\nu + \frac{1}{2}T \\ &= \tilde{T}_{\mu\nu}n^\mu n^\nu + \frac{1}{2}\tilde{T} + \epsilon(-\partial_\mu T \partial^\mu T) + \frac{1}{2}(-\epsilon). \end{aligned} \quad (58)$$

$$\frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} = \frac{\partial}{\partial T_0} \left( \frac{1}{s} \frac{\partial s}{\partial T_0} \right) + \left( \frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2. \quad (59)$$

We will use the commutator (38) and eqn. (41) to get

$$\begin{aligned} \frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} &= \frac{1}{3} \left\langle \frac{\partial_\mu T}{-\partial_\mu T \partial^\mu T} \partial^\mu \left( \frac{\theta}{(-\partial_\mu T \partial^\mu T)^{1/2}} \right) \right\rangle + \frac{1}{3} \left\langle \frac{\theta^2}{-\partial_\mu T \partial^\mu T} \right\rangle - \frac{3}{9} \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle^2 = \\ &= -\frac{1}{3} \left\langle \frac{\partial_\mu T \partial^\mu \theta}{(-\partial_\mu T \partial^\mu T)^{3/2}} \right\rangle + \frac{1}{3} \left\langle \frac{\theta^2}{-\partial_\mu T \partial^\mu T} \right\rangle - \frac{2}{9} \left\langle \frac{\theta}{(-\partial_\mu T \partial^\mu T)^{1/2}} \right\rangle^2 - \frac{1}{6} \left\langle \frac{\partial_\mu T \partial^\mu (\partial_\nu T \partial^\nu T)}{(-\partial_\mu T \partial^\mu T)^{5/2}} \right\rangle, \end{aligned} \quad (60)$$

rewriting

we can insert (58) into the first term

$$\partial^\mu T \partial_\mu \theta = \nabla_T \theta = -(-\partial_\mu T \partial^\mu T)^{1/2} n^\mu \nabla_\mu \theta, \quad (61)$$

$$\begin{aligned} \frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} &= \frac{1}{3} \left\langle \frac{\theta^2}{-\partial_\mu T \partial^\mu T} \right\rangle - \frac{2}{9} \left\langle \frac{\theta}{(-\partial_\mu T \partial^\mu T)^{1/2}} \right\rangle^2 - \frac{1}{6} \left\langle \frac{\partial_\mu T \partial^\mu (\partial_\mu T \partial^\mu T)}{(-\partial_\mu T \partial^\mu T)^{5/2}} \right\rangle + \\ &- \frac{1}{3} \left\{ 2 \left\langle \frac{\sigma^2}{-\partial_\mu T \partial^\mu T} \right\rangle + \frac{1}{3} \left\langle \frac{\theta^2}{-\partial_\mu T \partial^\mu T} \right\rangle - \left\langle \frac{\nabla_\mu (n^\mu \nabla_\mu n_\nu)}{-\partial_\mu T \partial^\mu T} \right\rangle \right\} + \frac{8\pi G_N}{3} \left\{ \left\langle \frac{\tilde{T}_{\mu\nu} n^\mu n^\nu + \frac{1}{2}\tilde{T}}{(-\partial_\mu T \partial^\mu T)^{1/2}} \right\rangle + \frac{1}{2} \left\langle \frac{\epsilon}{\partial_\mu T \partial^\mu T} + 2\epsilon \right\rangle \right\}. \end{aligned} \quad (62)$$

This is the effective Raychaudhuri equation for the domain. There is an analogy to the second Friedman equation, since it describes the acceleration of the domain dependent scale factor s.

## B. Evaluation in the ADM gauge

In this subsection we will evaluate the general equations in the ADM gauge, which was introduced above, and compare them to the effective equations obtained

by T. Buchert. As a reminder in the ADM gauge  $n_\mu = N(-1, 0, 0, 0)$  and  $n^\mu = \frac{1}{N}(1, -N^i)$  where  $N$  is the lapse and  $N^i$  the shift. Since we have chosen  $n_\mu$  to be orthogonal to the spatial hypersurfaces the shift vectors vanish i.e. the metric reads:  $ds^2 = -N^2 dt^2 + h_{ij} dx^i dx^j$ . Especially we get for

$$-\partial_\mu T \partial^\mu T = -\partial_0 T g^{00} \partial_0 T = \frac{1}{N^2} (\partial_0 T)^2, \quad (63)$$

and



$$\partial_0 T_0 = 1, \quad (64)$$

since  $T_0$  defines the hypersurface of constant physical time.

Given that, the Hamiltonian constraint reduces to

$$\begin{aligned} \left( \frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2 &= -\frac{1}{6} \langle R_s N^2 \rangle - \frac{1}{6} \left\{ \frac{2}{3} \left( \langle N^2 \theta^2 \rangle - \langle N \theta \rangle^2 \right) - 2 \langle N^2 \sigma^2 \rangle \right\} + \frac{8\pi G_N}{3} \langle N^2 \tilde{T}_{\mu\nu} n^\mu n^\nu + \epsilon \rangle = \\ &= -\frac{1}{6} \langle R_s N^2 \rangle - \frac{1}{6} Q_D + \frac{8\pi G_N}{3} \langle N^2 \tilde{T}_{\mu\nu} n^\mu n^\nu + \epsilon \rangle. \end{aligned} \quad (65)$$

And the Raychaudhuri equation reads as

$$\begin{aligned} -\frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} &= -\frac{2}{9} \left( \langle N^2 \theta^2 \rangle - \langle N \theta \rangle^2 \right) + \frac{2}{3} \langle N^2 \sigma^2 \rangle - \frac{1}{3} \langle \theta \partial_0 N \rangle - \frac{1}{3} \langle N h^{ij} \nabla_i \nabla_j N \rangle + \frac{8\pi G_N}{3} \left\langle N^2 \left( \tilde{T}_{\mu\nu} n^\mu n^\nu + \frac{1}{2} \tilde{T} \right) + \epsilon \right\rangle = \\ &= -\frac{1}{3} Q_D - \frac{1}{3} \langle \theta \partial_0 N \rangle - \frac{1}{3} \langle N h^{ij} \nabla_i \nabla_j N \rangle + \frac{8\pi G_N}{3} \left\langle \left( \tilde{T}_{\mu\nu} n^\mu n^\nu + \frac{1}{2} \tilde{T} \right) + \epsilon \right\rangle. \end{aligned} \quad (66)$$

Setting  $N = 1$  we obtain:

The Hamilton constraint density

$$\begin{aligned} \left( \frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2 &= -\frac{1}{6} \langle R_s \rangle - \frac{1}{6} \left\{ \frac{2}{3} \left( \langle \theta^2 \rangle - \langle \theta \rangle^2 \right) - 2 \langle \sigma^2 \rangle \right\} + \frac{8\pi G_N}{3} \langle \tilde{T}_{\mu\nu} n^\mu n^\nu + \epsilon \rangle = \\ &= -\frac{1}{6} \langle R_s \rangle - \frac{1}{6} Q_D + \frac{8\pi G_N}{3} \langle \tilde{T}_{\mu\nu} n^\mu n^\nu + \epsilon \rangle. \end{aligned} \quad (67)$$

And the Raychaudhuri equation:

$$\begin{aligned} -\frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} &= -\frac{2}{9} \left( \langle \theta^2 \rangle - \langle \theta \rangle^2 \right) + \frac{2}{3} \langle \sigma^2 \rangle + \frac{8\pi G_N}{3} \left\langle \left( \tilde{T}_{\mu\nu} n^\mu n^\nu + \frac{1}{2} \tilde{T} \right) + \epsilon \right\rangle \\ &= -\frac{1}{3} Q_D + \frac{8\pi G_N}{3} \left\langle \left( \tilde{T}_{\mu\nu} n^\mu n^\nu + \frac{1}{2} \tilde{T} \right) + \epsilon \right\rangle. \end{aligned} \quad (68)$$

Note: The coordinate dust energy  $\epsilon$  has a negative effect on the acceleration.

$$\langle S, \Omega \rangle = \frac{\int_{\Sigma_0} \sqrt{h(x)} S(x) \Theta(r_0 - B(x)) d^3 x}{\int_{\Sigma_0} \sqrt{h(x)} \Theta(r_0 - B(x)) d^3 x}. \quad (69)$$

The averaging functional with the window function reduces in this gauge to

### C. The average functional in the ADM gauge

To perform integration in general we need a push forward from the manifold into the  $\mathbb{R}^n$ , in our case a map from the dust space into  $\mathbb{R}^n$  is required. Define:  $\Psi := \Phi \circ X \circ Z^{-1}$  s.t.  $\Psi : S(T) \rightarrow \mathbb{R}^n$  where  $\Phi$  is the map  $\Phi : M \rightarrow \mathbb{R}^n$ ,  $Z^{-1}$  the map  $Z^{-1} : S(T) \rightarrow \Sigma$ ,  $X$  the map  $X : \Sigma \rightarrow M$  and  $S(T)$  the dust-space slice.

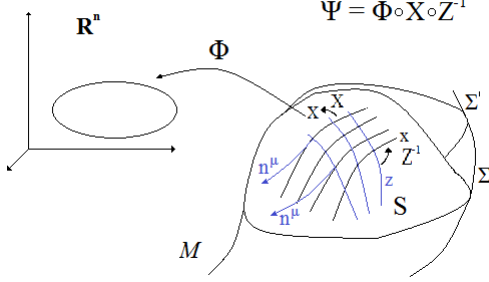


FIG. 2: The map  $\Psi$  used for the push forward into  $\mathbb{R}^n$ . This map is a combination of maps from the dust-space slice of constant dust-time to the space of internal variables:  $Z^{-1}$ , the map from this internal space to the manifold  $M$ :  $X$  and the map from the manifold to the  $\mathbb{R}^n$ :  $\Phi$ .

It was already shown that  $Z^2$  is gauge invariant so now it is legitimate to fix a gauge. A legitimate choice would be  $Z^{-1} = \mathbb{I}$  the identity,  $X$  the map from the hypersurface coordinates  $x^a = Z^i$  to the manifold coordinates  $X^\mu$  and therefore  $S = \Sigma$ . In the ADM gauge the coordinates are  $t$  and  $x^a$  as read off from the metric  $ds^2 = -dt^2 + h_{ab}dx^a dx^b$ . This leads to

$$Z^2(x) = Z^i h_{ij} Z^j = x^a h_{ab} x^b. \quad (70)$$

Rewriting the average functional one obtains

$$\begin{aligned} \langle S, \Omega \rangle &= \frac{\int_{\Sigma_0} \sqrt{h(x)} S(x) \Theta(r_0 - Z^2(x)) d^3x}{\int_{\Sigma_0} \sqrt{h(x)} \Theta(r_0 - Z^2(x)) d^3x} \\ &= \frac{\int_0^{r_0^2} \int_0^{2\pi} \int_0^\pi dr d\phi d\theta \sin(\theta) r^2 \sqrt{h} S(r, \phi, \theta)}{\int_0^{r_0^2} \int_0^{2\pi} \int_0^\pi dr d\phi d\theta \sin(\theta) r^2 \sqrt{h}}, \end{aligned} \quad (71)$$

which is a volume integral over a sphere of radius  $r_0^2$ . With this formalism the effect of inhomogeneity on an effective Friedmann model can be estimated and an example of application to the ideal fluid cosmology can be found in the Appendix A.

### D. Comparison to Buchert equations

The difference between our formalism and the averaging applied by T. Buchert in [3] is that we did not

choose a gauge from the beginning but deparametrized the manifold with dust. This introduced a time in the theory. Furthermore we found a way to perform gauge invariant averages over finite volumes. Hence the averaged quantities  $\{s \dot{s} \ddot{s} \langle \epsilon \rangle\}$  became physical observables and we obtained equations governing their dynamics. The equations we get after fixing the ADM gauge are slightly different to the ones in [3]. We recover Buchert's equations in our general formalism if we set the rest energy momentum  $\tilde{T}_{\mu\nu}$  to zero. Therefore one could think that in Buchert's formalism just the geometry is averaged. The problem with this point of view is that even an infinitesimal energy density  $\epsilon$  leads to a geometry arbitrary far away from the vacuum geometry, since there is no smooth limit  $\epsilon \rightarrow 0$ .

Hence one has to see that Buchert's equations describe correctly the evolution of a cosmology filled with one type of pressure-free fluid in the rest frame of this fluid. We see that in this special case the procedures of gauge fixing and averaging commute. Our formalism allows to generalize the averaging to an almost arbitrary energy momentum tensor. The only restriction is that the energy density must not vanish anywhere. Especially it is not possible to apply this formalism to space-time devoid of all matter.

## IV. SUMMARY

A manifestly gauge invariant averaging formalism for finite volumes was presented in this work. The question of gauge dependence of the backreaction was resolved. The model used just required the assumption of a non-vanishing energy density. In this case a dust energy momentum tensor can be separated from the total energy density. The coordinate fields of the dust were used as a reference frame to perform averaging.

The derived equations for the average of an arbitrary energy momentum tensor were evaluated in the ADM gauge, which corresponds to choosing the rest frame of the dust. Furthermore, as an example the equation is evaluated for an energy momentum tensor of an ideal fluid. The resulting equation was also studied in the ADM gauge. An interesting phenomenon was observed, as discussed in the ideal fluid cosmology in the Appendix A. It appears that "tilt" effects (i.e. non-co-linearity of the dust flow and the fluid's flow) contribute negatively to the acceleration of the domain dependent scale factor.

The derived equations in the ADM gauge were compared to the Buchert equations. We are able to interpret the Buchert equations in the gauge independent framework. It turned out that they describe the averaged behaviour of a pressure-free fluid, using this fluid itself as a reference and choosing its rest frame. It turns out that in this special case gauge fixing and averaging commute.

By deriving gauge independent equations for the cosmological backreaction, we proved that it is in principle

an observable. With this formalism at hand it is possible to study cosmological data and can trust the derived equations to correctly describe the observations, once we identify a system of freely falling observers.

One could go one step further and argue that given the necessity to explain the abundance of a matter species, which mainly -or entirely- interacts gravitationally, we can view our system with the coordinate dust as a realistic cosmological model. In which case the dust is the Dark Matter which behaves, according to observations as a pressure-less fluid.

An interesting and may be fundamental observation is, that it is not possible to define averaged observables in a space-time devoid of matter.

### Acknowledgments

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### Appendix A: Example of an Ideal fluid cosmology

A cosmology equipped with an energy momentum tensor of an ideal fluid will be studied. This is a reasonable choice for the energy momentum, since it is the most general tensor fulfilling the symmetries of an isotropic universe and since we know from the CMB that the universe has been extremely isotropic this model is a reasonable choice. This procedure can be easily generalized to an arbitrary number of ideal non interacting fluids.

### Normal projection of the equations of motion

The Einstein tensor multiplied with the time flow vectors (which is equivalent to the Hamiltonian constraint density) gives:

$$G^{\mu\nu}n_\mu n_\nu = \frac{1}{2}R_s + \frac{1}{3}\theta^2 - \sigma^2, \quad (\text{A1})$$

where the expansion rate and the shear are defined w.r.t  $n_\mu$ , the  $R_s$  is the spatial curvature associated with the projector on the hypersurface orthogonal to  $n_\mu$ . On the other hand one has from the matter contribution

$$T^{\mu\nu}n_\mu n_\nu = (T_{\text{fluid}}^{\mu\nu} + T_{\text{dust}}^{\mu\nu})n_\mu n_\nu, \quad (\text{A2})$$

with

$$\begin{aligned} T_{\text{fluid}}^{\mu\nu}n_\mu n_\nu &= (\rho + p)(u^\mu n_\mu)^2 - p \\ &= \rho + (\rho + p) \sinh^2(\alpha_T) \end{aligned} \quad (\text{A3})$$

and

$$\begin{aligned} T_{\text{dust}}^{\mu\nu}n_\mu n_\nu &= \epsilon(v_\mu n^\mu)^2 = \epsilon(-\partial_\mu T n^\mu)^2 \\ &= \epsilon(-\partial_\mu T \partial^\mu T), \end{aligned} \quad (\text{A4})$$

where the tilt angle  $\alpha_T$  defined via  $\sinh^2(\alpha_T) := (u^\mu n_\mu)^2 - 1$ . Furthermore as discussed above  $v^\mu = \sqrt{-\partial_\mu T \partial^\mu T} n^\mu$  since we said that the velocity field  $W^k$  vanishes. Now the Hamiltonian constraint can be averaged. For normalization divide it first by  $-3\partial_\mu T \partial^\mu T$  and apply the averaging functional introduced above

$$\begin{aligned} &\left\langle \frac{1}{6} \frac{R_s}{-\partial_\mu T \partial^\mu T} + \frac{1}{9} \frac{\theta^2}{-\partial_\mu T \partial^\mu T} - \frac{1}{3} \frac{\sigma^2}{-\partial_\mu T \partial^\mu T} \right\rangle + \frac{1}{9} \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle^2 - \frac{1}{9} \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle^2 = \\ &= \frac{8\pi G_N}{3} \left\langle \frac{\rho + (\rho + p) \sinh^2(\alpha_T)}{-\partial_\mu T \partial^\mu T} + \epsilon \right\rangle. \end{aligned} \quad (\text{A5})$$

As above we rewrite (A5) as

$$\left( \frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2 = -\frac{1}{6} \left\langle \frac{R_s}{-\partial_\mu T \partial^\mu T} \right\rangle - \frac{1}{6} Q_D + \frac{8\pi G_N}{3} \left\langle \frac{\rho + (\rho + p) \sinh^2(\alpha_T)}{-\partial_\mu T \partial^\mu T} + \epsilon \right\rangle \quad (\text{A6})$$

This is the effective equation describing the velocity

of the expansion of our domain-dependent scale factor.

With the definitions as above we can write the cosmological balance equation as:

$$H_D := \frac{1}{s} \frac{\partial s}{\partial T_0}, \quad (\text{A7})$$

$$\Omega_m := \frac{8\pi G_N}{3H_D^2} \left\langle \frac{\rho + (\rho + p) \sinh^2(\alpha_T)}{-\partial_\mu T \partial^\mu T} \right\rangle, \quad (\text{A8})$$

$$\Omega_\epsilon := \frac{8\pi G_N}{3H_D^2} \langle \epsilon \rangle, \quad (\text{A9})$$

$$\Omega_k := -\frac{\langle R \rangle}{6H_D^2}, \quad (\text{A10})$$

$$\Omega_Q := -\frac{Q_D}{6H_D^2}, \quad (\text{A11})$$

$$\Omega_k + \Omega_Q + \Omega_\epsilon + \Omega_m = 1. \quad (\text{A12})$$

From the energy balance equation we see that the back-reaction term  $\Omega_Q$  opens the possibility of contributing to an effective cosmological constant term in this model, which is in principle close to  $\Lambda$ -CDM.

*Raychaudhuri equation:*

Combining the Hamiltonian constraint with the trace of the Einstein equations projected orthogonal to the time flow, one gets:

$$R_{\mu\nu} n^\mu n^\nu = T_{\mu\nu} h^{\mu\nu} - \frac{1}{2} T \quad (\text{A13})$$

Which is equivalent to

$$\begin{aligned} -n^\mu \nabla_\mu \theta &= 2\sigma^2 + \frac{1}{3} \theta^2 - \nabla^\nu (n^\mu \nabla_\mu n_\nu) \\ &+ (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) n^\mu n^\nu. \end{aligned} \quad (\text{A14})$$

In our model the last term gives

$$(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) n^\mu n^\nu = T_{\mu\nu}^{\text{fluid}} n^\mu n^\nu + T_{\mu\nu}^{\text{dust}} n^\mu n^\nu + \frac{1}{2} T \quad (\text{A15})$$

$$= \rho + \sinh^2(\alpha_T)(\rho + p) + \epsilon(-\partial_\mu T \partial^\mu T) + \frac{1}{2}(3p - \rho - \epsilon).$$

Again we will use the commutator (38) and Eqn. (41) to get

$$\begin{aligned} \frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} &= \frac{1}{3} \left\langle \frac{\partial_\mu T}{-\partial_\mu T \partial^\mu T} \partial^\mu \left( \frac{\theta}{(-\partial_\mu T \partial^\mu T)^{1/2}} \right) \right\rangle + \frac{1}{3} \left\langle \frac{\theta^2}{-\partial_\mu T \partial^\mu T} \right\rangle - \frac{3}{9} \left\langle \frac{\theta}{\sqrt{-\partial_\mu T \partial^\mu T}} \right\rangle^2 = \\ &= -\frac{1}{3} \left\langle \frac{\partial_\mu T \partial^\mu \theta}{(-\partial_\mu T \partial^\mu T)^{3/2}} \right\rangle + \frac{1}{3} \left\langle \frac{\theta^2}{-\partial_\mu T \partial^\mu T} \right\rangle - \frac{2}{9} \left\langle \frac{\theta}{(-\partial_\mu T \partial^\mu T)^{1/2}} \right\rangle^2 - \frac{1}{6} \left\langle \frac{\partial_\mu T \partial^\mu (\partial_\nu T \partial^\nu T)}{(-\partial_\mu T \partial^\mu T)^{5/2}} \right\rangle. \end{aligned} \quad (\text{A16})$$

Rewriting

we can insert (A14) into the first term

$$\partial^\mu T \partial_\mu \theta = \nabla_T \theta = -(-\partial_\mu T \partial^\mu T)^{1/2} n^\mu \nabla_\mu \theta, \quad (\text{A17})$$

$$\begin{aligned} \frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} &= \frac{1}{3} \left\langle \frac{\theta^2}{-\partial_\mu T \partial^\mu T} \right\rangle - \frac{2}{9} \left\langle \frac{\theta}{(-\partial_\mu T \partial^\mu T)^{1/2}} \right\rangle^2 - \frac{1}{6} \left\langle \frac{\partial_\mu T \partial^\mu (\partial_\nu T \partial^\nu T)}{(-\partial_\mu T \partial^\mu T)^{5/2}} \right\rangle + \frac{1}{3} \left\{ 2 \left\langle \frac{\sigma^2}{-\partial_\mu T \partial^\mu T} \right\rangle \right. \\ &\left. + \frac{1}{3} \left\langle \frac{\theta^2}{-\partial_\mu T \partial^\mu T} \right\rangle - \left\langle \frac{\nabla_\mu (n^\mu \nabla_\nu n^\nu)}{-\partial_\mu T \partial^\mu T} \right\rangle \right\} + \frac{8\pi G_N}{3} \left\{ \left\langle \frac{(\rho + p) \sinh^2(\alpha_T) + \rho}{(-\partial_\mu T \partial^\mu T)^{1/2}} \right\rangle + \frac{1}{2} \left\langle \frac{3p - \rho - \epsilon}{-\partial_\mu T \partial^\mu T} + 2\epsilon \right\rangle \right\}. \end{aligned} \quad (\text{A18})$$

This is the effective Raychaudhuri equation for the do-

main in the case of the energy momentum tensor being

that of an ideal fluid.

The Hamiltonian constraint density reduces to

### 1. Evaluation in the ADM gauge

In this subsection we will evaluate the equations of 4.2.5 in the ADM gauge, which was introduced above.

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$$\begin{aligned} \left( \frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2 &= -\frac{1}{6} \langle R_s N^2 \rangle - \frac{1}{6} \left\{ \frac{2}{3} \left( \langle N^2 \theta^2 \rangle - \langle N \theta \rangle^2 \right) - 2 \langle N^2 \sigma^2 \rangle \right\} + \frac{8\pi G_N}{3} \langle N^2 (\rho + (\rho + p) \sinh^2(\alpha_T)) + \epsilon \rangle = \\ &= -\frac{1}{6} \langle R_s N^2 \rangle - \frac{1}{6} Q_D + \frac{8\pi G_N}{3} \langle N^2 (\rho + (\rho + p) \sinh^2(\alpha_T)) + \epsilon \rangle . \end{aligned} \quad (\text{A19})$$


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And the Raychaudhuri equation to

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$$\begin{aligned} -\frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} &= -\frac{2}{9} \left( \langle N^2 \theta^2 \rangle - \langle N \theta \rangle^2 \right) + \frac{2}{3} \langle N^2 \sigma^2 \rangle - \frac{1}{3} \langle \theta \partial_0 N \rangle - \frac{1}{3} \langle N h^{ij} \nabla_i \nabla_j N \rangle + \\ &\quad + \frac{4\pi G_N}{3} \langle 2 N^2 ((\rho + p) \sinh^2(\alpha_T)) + N^2 (\rho + 3p) + 2\epsilon \rangle \\ &= -\frac{1}{3} Q_D - \frac{1}{3} \langle \theta \partial_0 N \rangle - \frac{1}{3} \langle N h^{ij} \nabla_i \nabla_j N \rangle + \frac{4\pi G_N}{3} \langle 2 N^2 ((\rho + p) \sinh^2(\alpha_T)) + N^2 (\rho + 3p) + 2\epsilon \rangle . \end{aligned} \quad (\text{A20})$$


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Setting  $N = 1$  we obtain:

The Hamilton constraint density

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$$\begin{aligned} \left( \frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2 &= -\frac{1}{6} \langle R_s \rangle - \frac{1}{6} \left\{ \frac{2}{3} \left( \langle \theta^2 \rangle - \langle \theta \rangle^2 \right) - 2 \langle \sigma^2 \rangle \right\} + \frac{8\pi G_N}{3} \langle (\rho + (\rho + p) \sinh^2(\alpha_T)) + \epsilon \rangle \\ &= -\frac{1}{6} \langle R_s \rangle - \frac{1}{6} Q_D + \frac{8\pi G_N}{3} \langle \rho + (\rho + p) \sinh^2(\alpha_T) + \epsilon \rangle . \end{aligned} \quad (\text{A21})$$


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And the Raychaudhuri equation

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$$\begin{aligned} -\frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} &= -\frac{2}{9} \left( \langle \theta^2 \rangle - \langle \theta \rangle^2 \right) + \frac{2}{3} \langle \sigma^2 \rangle + \frac{8\pi G_N}{3} \langle 2 ((\rho + p) \sinh^2(\alpha_T)) + (\rho + 3p) + 2\epsilon \rangle = \\ &= \frac{8\pi G_N}{3} \langle 2 ((\rho + p) \sinh^2(\alpha_T)) + (\rho + 3p) + 2\epsilon \rangle - \frac{1}{3} Q_D . \end{aligned} \quad (\text{A22})$$


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Observations:

- Both the Hamiltonian constraint and the Ray-

chaudhuri equation contain the energy density of the coordinate dust. In the second equation it contributes negatively to the acceleration of the scale factor.

- The fact that generically the flow of the coordinate dust is non parallel to the flow of the fluid which is averaged, is manifested in the  $\sinh^2(\alpha_T)$  term. This as well contributes negatively to the acceleration of the scale factor  $s$ .

- After the gauge choice the spatial integral is an ordinary 3-dimensional integral.
- The backreaction  $Q_D$  has the potential to contribute to the acceleration. This has been addressed in the literature [11–13] and certainly deserves attention.

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| <p>[1] T. Buchert, Gen.Rel.Grav. <b>33</b>, 1381 (2001), gr-qc/0102049.</p> <p>[2] T. Buchert (1999), gr-qc/0001056.</p> <p>[3] T. Buchert, Gen.Rel.Grav. <b>32</b>, 105 (2000), gr-qc/9906015.</p> <p>[4] M. Gasperini, G. Marozzi, and G. Veneziano, JCAP <b>1002</b>, 009 (2010), 0912.3244.</p> <p>[5] G. Marozzi, JCAP <b>1101</b>, 012 (2011), 1011.4921.</p> <p>[6] C. Rovelli (2004).</p> <p>[7] B. Dittrich, Class.Quant.Grav. <b>23</b>, 6155 (2006), gr-qc/0507106.</p> | <p>[8] K. Giesel, S. Hofmann, T. Thiemann, and O. Winkler, Class.Quant.Grav. <b>27</b>, 055006 (2010), 0711.0117.</p> <p>[9] T. Thiemann (2006), astro-ph/0607380.</p> <p>[10] R. L. Arnowitt, S. Deser, and C. W. Misner, Gen.Rel.Grav. <b>40</b>, 1997 (2008), gr-qc/0405109.</p> <p>[11] S. Rasanen, Class.Quant.Grav. <b>28</b>, 164008 (2011), 1102.0408.</p> <p>[12] N. Li, M. Seikel, and D. J. Schwarz, Fortsch.Phys. <b>56</b>, 465 (2008), 0801.3420.</p> <p>[13] T. Buchert, Class.Quant.Grav. <b>28</b>, 164007 (2011), 1103.2016.</p> |
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